Facts about Improper Integrals	**Explanation**
What is an Improper Integral? When does an Improper Integral Converge or Diverge?	An improper integral is one in which the interval of integration is unbounded or the integrand has a discontinuity at some point in the interval of integration. If the value of the improper integral is a finite number, then the improper integral converges. Otherwise, it diverges.
Rewriting an Improper Integral using Limits in Order to Compute its Value.	Let a,b be any number in the interval of integration (including the endpoints) and c be a discontinuous point for $f(x)$ within the interval of integration. Case 1: $$\int_{-\infty}^{b} f(x)\ dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\ dx$$ Case 2: $$\int_{a}^{\infty} f(x)\ dx = \lim_{t\to\infty}\int_{a}^{t} f(x)\ dx$$ Case 3: $$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{a} f(x)\ dx + \int_{a}^{\infty} f(x)\ dx$$ Case 4: $$\int_{a}^{c} f(x)\ dx = \lim_{t\to c^-}\int_{a}^{t} f(x)\ dx$$ Case 5: $$\int_{c}^{b} f(x)\ dx = \lim_{t\to c^+}\int_{t}^{b} f(x)\ dx$$ Case 6: $$\int_{a}^{b} f(x)\ dx = \int_{a}^{c} f(x)\ dx + \int_{c}^{b} f(x)\ dx$$

1. Consider the improper integral $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$.

   (a) Are there any discontinuities for the function $f(x) = \frac{e^x}{1+e^{2x}}$ within the range of the limits of integration?

   Solution:
   There are no discontinuities.

   (b) Rewrite the improper integral using limit notation. Note that it may be easier to use $x = 0$ as a reference point.

   Solution:
   $$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{t\to-\infty} \int_t^0 \frac{e^x}{1+e^{2x}}\ dx + \lim_{t\to\infty} \int_0^t \frac{e^x}{1+e^{2x}}\ dx$$

   (c) Evaluate the improper integral.

   Solution:
   $$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx &= \lim_{t\to-\infty} \int_t^0 \frac{e^x}{1+e^{2x}}\ dx + \lim_{t\to\infty} \int_0^t \frac{e^x}{1+e^{2x}}\ dx \\ &= \lim_{t\to-\infty} \left[arctan(e^x)\right]_t^0 + \lim_{t\to\infty} \left[arctan(e^x)\right]_0^t \\ &= \lim_{t\to-\infty} \left[arctan(e^0) - arctan(e^t)\right] + \lim_{t\to\infty} \left[arctan(e^t) - arctan(e^0)\right] \\ &= \lim_{t\to-\infty} \left[-arctan(e^t)\right] + \lim_{t\to\infty} \left[arctan(e^t)\right] \\ &= -0 + \frac{\pi}{2}\end{aligned}$$

   (d) Does the improper integral converge or diverge?

   Solution:
   We obtain a finite number, so the improper integral converges.

2. Consider the improper integral $\int_0^{\infty} e^{-x} dx$.

   (a) Are there any discontinuities for the function $f(x) = e^{-x}$ within the range of the limits of integration?

Solution:
There are no discontinuities.

(b) Rewrite the improper integral using limit notation.

Solution:

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx$$

(c) Evaluate the improper integral.

Solution:

$$\begin{aligned}\int_0^\infty e^{-x}dx &= \lim_{t\to\infty}\int_0^t e^{-x}\,dx \\ &= \lim_{t\to\infty}\left[-e^{-x}\right]_0^t \\ &= \lim_{t\to\infty} -e^{-t} + e^{-0} \\ &= \lim_{t\to\infty} -\frac{1}{e^t} + 1 \\ &= 1\end{aligned}$$

(d) Does the improper integral converge or diverge?

Solution:
We obtain a finite number, so the improper integral converges.

3. Consider the improper integral $\int_0^1 \frac{1}{\sqrt[3]{x}} dx$.

   (a) Are there any discontinuities for the function $f(x) = \frac{1}{\sqrt[3]{x}}$ within the range of the limits of integration?

   Solution:
   There is a discontinuity at $x = 0$.

   .

   (b) Rewrite the improper integral using limit notation.

   Solution:

$$\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt[3]{x}} \, dx$$

   (c) Evaluate the improper integral.

   Solution:

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt[3]{x}} dx &= \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt[3]{x}} \, dx \\ &= \lim_{t \to 0^+} \left[\frac{3x^{\frac{2}{3}}}{2} \right]_t^1 \\ &= \lim_{t \to 0^+} \frac{3}{2} - \frac{3}{2} t^{\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

   (d) Does the improper integral converge or diverge?

   Solution:
   We obtain a finite number, so the improper integral converges.

4. What happens when you evaluate $\displaystyle\int_0^\infty \frac{1}{x}\,dx$? How about $\displaystyle\int_1^\infty \frac{1}{x}\,dx$? And $\displaystyle\int_0^1 \frac{1}{x}\,dx$?

Solution:

Notice that our function $1/x$ blows up to infinity when $x \to 0$, so we actually need to split the integral into two integrals and take two limits. We'll choose $x = 1$ as the point to split at.

$$\begin{aligned}
\int_0^\infty \frac{1}{x}\,dx &= \int_0^1 \frac{1}{x}\,dx + \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx + \lim_{s\to\infty} \int_1^s \frac{1}{x}\,dx \\
&= \lim_{t\to 0^+} \ln|x|\Big|_t^1 + \lim_{s\to\infty} \ln|x|\Big|_1^s \\
&= \ln(1) - \lim_{t\to 0+} \ln(t) + \lim_{s\to\infty} \ln(s) - \ln(1) \\
&= \infty,
\end{aligned}$$

so the integral diverges. Notice that in doing taking this integral, we actually did the two other integrals the problem asks for as well, and found that both of them diverge as well. Of all functions of the form $1/x^p$ for $p > 0$, the function $1/x$ is the only one that diverges going to 0 and to ∞!